A Law of Large Numbers for Betting Systems

A betting system is a set of rules that determines what bet to make given the results of all previous trials. This could include not only the amount to be bet, but also the type of bet.

Let X_M be the gain on the Mth trial. Let $E(X_M | R_{M-1})$ be the expected gain on the Mth trial calculated after the results of the previous M - 1 trials are known. Since the bet on the Mth trial may vary depending on the outcome of the previous trials, $E(X_M | R_{M-1})$ is a random variable. $x_1, x_2, x_3, \ldots$ are the possible values that X_M may take on. r_j is the event that the jth possible outcome of the first M - 1 trials has occurred. r_j stands not just for a particular outcome of the M-1th trial but for a particular outcome of the entire sequence of trials from the first through the M-1th. These particular outcomes can be numbered 1, 2, 3, . . . For example, the possible results of the first M-1 spins of a roulette wheel with 38 numbers can be assigned the numbers 1, 2, 3, . . . , 38^{M-1} . $E(X_M | r_j)$ is the expected gain on the Mth trial calculated knowing that the outcome of the first M-1 trials was r_j.

$$E(X_M | r_j) = \sum_i x_i p(X_M = x_i | r_j) \quad \text{Note that} \quad \sum_i p(X_m = x_i | r_j) = 1.$$

We now calculate that $E(X_M - E(X_M | R_{M-1})) = 0$.

$$E(X_M - E(X_M | R_{M-1})) = \sum_j \sum_i (x_i - E(X_M | r_j)) p(X_M = x_i | r_j) p(r_j)$$

$$= \sum_j p(r_j) \left[\left(\sum_i x_i p(X_M = x_i | r_j) - E(X_M | r_j) \sum_i p(X_M = x_i | r_j) \right] \right.$$

$$= \sum_j p(r_j) \left[E(X_M | r_j) - E(X_M | r_j) \right] = 0.$$

Let $K < M$. The value that $E(X_K - E(X_K | R_{K-1}))$ takes on is completely determined once the particular result of the first M-1 trials is known because that result also tells us the result of the first K trials. So let r_j be a particular result of the first M-1 trials and let C_j be the value that $E(X_K - E(X_K | R_{K-1}))$ takes on given that r_j occurred. We now calculate that
$E[(X_K - E(X_K | R_{K-1}))(X_M - E(X_M | R_{M-1}))] = 0.$

$E[(X_K - E(X_K | R_{K-1}))(X_M - E(X_M | R_{M-1}))] =$

$$\sum_j \sum_i C_j (x_i - E(X_M | r_j)) p(X_M = x_i | r_j) p(r_j) \quad =$$

$$\sum_j p(r_j) C_j [\sum_i x_i p(X_M = x_i | r_j) - E(X_M | r_j) \sum_i p(X_M = x_i | r_j)] =$$

$$\sum_j p(r_j) C_j [E(X_M | r_j) - E(X_M | r_j)] = 0.$$

The above calculations show that the expectations and covariances of the random variables $X_i - E(X_i | R_{i-1})$ are all zero.

Chebysev's Inequality

The probability that $|Y - E(Y)| <= \varepsilon$ is greater than $1 - V(Y)/\varepsilon^2$.

Let Y =1/n $\sum_{i=1}^{i=n}(X_i - E(X_i|R_{i-1}))$, then since $E(X_i - E(X_i | R_{i-1})) = 0$, $E(Y) = 0$, and since the covariances of the $(X_i - E(X_i | R_{i-1}))$'s are all zero

$V(Y) = 1/n^2 \sum_{i=1}^{i=n} V(X_i - E(X_i | R_{i-1}))$. If there is a finite number D such that the maximum amount that can be won or lost on any trial < D, then there is a number C such that for every i,
$V(X_i - E(X_i | R_{i-1})) < C$. In that case, $V(Y) < nC/n^2 = C/n$.
Substituting into Chebyshev's Inequality gives:

The probability that $|1/n \sum_{i=1}^{i=n}(X_i - E(X_i|R_{i-1}))| <= \varepsilon$ is greater than

$1\text{-}C/n\varepsilon^2$.

So if there is a finite number D such that the maximum amount that can be won or lost on any trial < D, then as n approaches infinity, the probability that $|1/n\sum_{i=1}^{i=n} X_i - 1/n\sum_{i=1}^{i=n} E(X_i | R_{i-1})| <= \varepsilon$ approaches one.

Let the random variable B_i be the amount bet on the ith trial, then $\sum_{i=1}^{i=n} B_i$ = the total amount bet in n trials. If there is a minimum bet of 1 unit on every trial, then $\sum_{i=1}^{i=n} B_i$ >= n and then whenever

$|1/n\sum_{i=1}^{i=n} X_i - 1/n\sum_{i=1}^{i=n} E(X_i \mid R_{i-1})| <= \varepsilon$,

$|\sum_{i=1}^{i=n} X_i/\sum_{i=1}^{i=n} B_i - \sum_{i=1}^{i=n} E(X_i \mid R_{i-1})/\sum_{i=1}^{i=n} B_i| <= \varepsilon$. So in this case, if the

probability that $|1/n\sum_{i=1}^{i=n} X_i - 1/n\sum_{i=1}^{i=n} E(X_i \mid R_{i-1})| <= \varepsilon$ approaches one as n approaches infinity, then the probability that

$|\sum_{i=1}^{i=n} X_i/\sum_{i=1}^{i=n} B_i - \sum_{i=1}^{i=n} E(X_i \mid R_{i-1})/\sum_{i=1}^{i=n} B_i| <= \varepsilon$ will also approach one as n approaches infinity.

We have proved a Law of Large Numbers for Betting Systems:

If there is a minimum bet of 1 unit on each trial and if there is a finite number D such that the maximum amount that can be won or lost on any trial < D, then no matter how the player varies the bets based on the results of all previous trials, the probability that

$|\sum_{i=1}^{i=n} X_i/\sum_{i=1}^{i=n} B_i - \sum_{i=1}^{i=n} E(X_i \mid R_{i-1})/\sum_{i=1}^{i=n} B_i| <= \varepsilon$ will approach one as n approaches infinity.

Let v be the value that the random variable $\sum_{i=1}^{i=n} E(X_i \mid R_{i-1}) / \sum_{i=1}^{i=n} B_i$ takes on for a particular outcome of the first n trials.

$v = (c_1b_1+c_2b_2+ . . . +c_nb_n)/(b_1+b_2+ . . . +b_n)$

where c_i is the expected gain per unit bet for the actual type of bet made on the ith trial and where b_i is the amount of the actual bet made on the ith trial.

v is a weighted average of the c's. If the possible values that the c's can take on are: -.01, -.05, and -.1 then v cannot be less negative than -.01 nor more negative than -.1.
The probability statement says that if n is sufficiently large, the probability will be very high that the actual gain per unit bet will be very close to v. So in the example above, no matter what the player does, the probability will be very high that he or she will have lost at least close to -.01 of the total amount bet. In blackjack some of the c's will be positive and some will be negative. By counting cards the player can tell when the c's are positive and when they are negative. By making large bets when the c's are positive and small bets when the c's are negative, v will be positive and the player has a high probability of having a positive gain per unit bet after a sufficiently large number of trials. In certain situations, the c's will all be the same. In that case $v = (cb_1+cb_2+ . . . +cb_n)/(b_1+b_2+ . . . +b_n) = c$.

For example, if the gambler is playing las vegas style roulette and is betting on the color red, the expected gain per unit bet is:

$+1 \times 18/38 + -1 \times 20/38 = -1/19$

So no matter how the gambler varies the amount bet based on the previous results, if he or she has to keep the bets between a minimum and a maximum then the probability is very high that after a very large number of trials the gambler will have lost close to 1/19 of the total amount bet.

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